ALGEBRAIC CRITERION FOR STOCHASTIC STABILITY OF LINEAR SYSTEMS WITH PARAMETRIC ACTION OF THE WHITE NOISE TYPE

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A class of linear systems of differential equations of Ito is examined. The algebraic criterion for exponential stability in the mean square is given. This criterion is easily applied in the case where the system is given by a transfer matrix.

The stability of stochastic differential equations was fist examined in papers [1, 2]. Linear systems of differential equations of Ito were studied in detail in papers [3 - 8]. The necessary and sufficient condition for stochastic exponential stability in the mean square can be found in papers [3, 8] for such systems. This condition consists of the fact that the spectrum of some square matrix, which is constructed with respect to parameters of the system, lies in the open left half-plane. In practice a check of this condition for a system of the order ν is reduced to the computation of no less than $\nu(\nu+1)/2$ determinants of the order $1, 2, \ldots, \nu(\nu+1)/2$. The execution of this procedure becomes difficult for large ν The special form of systems, which are examined in this paper and which are characteristic for a large number of applied problems, permits to establish another, more convenient criterion for stability.

1. Formulation of the problem. Let us examine the system of linear differential equations of Ito

$$x^{\cdot} = Px + \sum_{l=1}^{k} q_l \varphi_l, \qquad \sigma_l = r_l^* x \tag{1.1}$$

$$\varphi_l = \alpha_l \xi_l^{\cdot} \sigma_l \qquad (l = 1, 2, \ldots, k) \tag{1.2}$$

Here φ_l, σ_l, α_l are scalar quantities; x, q_l, r_l are vectors of the dimension ν. The matrix P has the dimension $\nu \times \nu$. Parameters of the system α_l, q_l, r_l and P are constant. It is assumed that $\xi_1, \xi_2, \ldots, \xi_k$ are scalar independent Gaussian white noises with unit spectral density [3]. The real quantities α_l have the meaning of noise intensities perturbing the system. The asterisk indicates the operation of matrix transposition and complex conjugation of its elements.

The following definition of stability of Eqs. (1.1), (1.2) introduced first in papers [1, 2] is used below.

Definition 1. System (1.1), (1.2) is called stochastically exponentially stable in the mean square if positive values A and ε exist such that for any $t \geqslant t_0$ and any ν-dimensional vector x_0 the inequality $M|x(t)|^2 \leqslant A|x|^2 \exp(-\varepsilon(t-t_0))$, holds. Here $x(t)$ is the solution of system (1.1), (1.2) determined by the condition $x(t_0) = x_0$. The symbol M represents mathematical expectation. For the sake of brevity the stability of the stochastic system in the sense of Definition 1 will be called simply stability.

In applications systems of differential equations are frequently given in the form

$$\sigma = -\chi(p)\varphi,\ \sigma = \|\sigma_l\|,\ \varphi = \|\varphi_l\| \qquad (l = 1, 2, \ldots, k) \tag{1.3}$$

Here p is the operator of differentiation with time, $\chi(\lambda)$ is the transfer matrix of the system from inputs φ_m to outputs σ_l. The elements of this matrix are proper rational functions of complex parameter λ. It is known that from a given matrix $\chi(\lambda)$ of proper rational functions it is possible to construct a matrix P and vectors q_m, r_l such that the relationships

$$\chi(\lambda) = \|\chi_{lm}(\lambda)\|_{l,\, m=1}^{k}, \qquad \chi_{lm} = r_l^*(P - \lambda I)^{-1} q_m \tag{1.4}$$

are satisfied and system (1.1), (1.2) is completely controllable and observable. Any such system (1.1), (1.2) will be called the normal form of system (1.3), (1.2). As is customary, by solutions of system (1.3), (1.2) below we mean solutions $\sigma_l(t)$ corresponding to the normal form (1.1), (1.2).

Definition 2. System (1.3), (1.2) is called stable if its normal form is stable. We note that the procedure for the determination of matrix P and vectors q_m and r_l from the transfer matrix $\chi(\lambda)$ of the system is frequently difficult, particularly in the case of multiple poles $\chi(\lambda)$. Therefore it is desirable to obtain the stability conditions for system (1.3), (1.2) in terms of the transfer matrix. Using results of papers [3, 4] it is easy to show that the stability of the system is preserved when the noise intensities are lowered. More exactly speaking, if system (1.3), (1.2) is stable for values of noise intensities $\alpha_1^\circ, \ldots, \alpha_k^\circ$, then it will remain stable for any intensities $\alpha_1, \ldots, \alpha_k$ which satisfy the conditions $|\alpha_l| \leqslant |\alpha_l^\circ|$ $(l = 1, 2, \ldots, k)$.

Definition 3. The vector of intensities $\alpha^\circ = \|\alpha_l^\circ\|$ is called critical for the system (1.3), (1.2) or its normal representation (1.1), (1.2) if the system is stable for all vectors $\varepsilon\alpha^\circ$ for $0 \leqslant \varepsilon < 1$ and unstable for vectors of intensities $\varepsilon\alpha^\circ$ for $\varepsilon \geqslant 1$. If for zero intensities $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$ the determinate system (1.1), (1.2) is asymptotically stable, then system (1.3), (1.2) is stable for sufficiently small intensities [3, 4]. Consequently, in the space of parameters $\alpha_1, \ldots, \alpha_k$ the critical intensities separate the region of stability of system (1.3), (1.2). In tne case of single-parameter perturbation $(k = 1)$ the interval $(-\alpha^\circ, \alpha^\circ)$ serves as the region of stability of system (1.3), (1.2). Here the quantity α° represents the critical intensity of noise.

2. Fundamental results. Let us assume initially that the normal form of system (1.1), (1.2) is given. We shall consider P to be a Hurvitz matrix, i.e. its spectrum lies in the open left half-plane. This condition is necessary for stability. Let us denote by A the linear operator associating to each of the $\nu \times \nu$ matrices G the matrix $A(G) = H$, where H is the unique solution of the matrix equation

$$P^*H + HP = -G \tag{2.1}$$

It is known that if G is real, symmetrical, positive definite or semi-definite, then $H = A(G)$ also have the same properties.

From coefficients of system (1.1), (1.2) we compose the $k \times k$-dimensional matrix

$$R = \|\alpha_l^2 \rho_{lm}\|_{lm=1}^{k}, \qquad \rho_{lm} = q_m^* A(r_l r_l^*)\, q_m \tag{2.2}$$

From previous statements it follows that elements R are nonnegative. We can show that $\rho_{lm} = 0$ when, and only when $\chi_{lm}(\lambda) \equiv 0$.

Theorem 1. System (1.1), (1.2) is stable when, and only when the P-matrix of Hurvitz and the eigenvalues R in modulus are smaller than unity.

The matrix R can be determined directly from the transfer matrix of the system $\chi(\lambda)$.

Lemma 1. Let $\chi_{lm}(\lambda) = \gamma_{lm}(\lambda)/\Delta_{lm}(\lambda)$, where $\Delta_{lm}(\lambda)$ is the Hurvitz polynomial. Then the equation

$$\chi_{lm}(\lambda)\,\chi_{lm}(-\lambda) = \frac{\tau_{lm}(\lambda)}{\Delta_{lm}(\lambda)} + \frac{\tau_{lm}(-\lambda)}{\Delta_{lm}(-\lambda)} \tag{2.3}$$

has as its solution the polynomial $\tau_{lm}(\lambda)$ the degree of which is lower than the degree of $\Delta_{lm}(\lambda)$. Such a solution is unique. The matrix R is defined by equations

$$R = \| \alpha_l^2 \rho_{lm} \|_{l,m=1}^{k}, \qquad \rho_{lm} = \lim_{|\lambda|\to\infty} \lambda \frac{\tau_{lm}(\lambda)}{\Delta_{lm}(\lambda)} \tag{2.4}$$

Theorem 2. System (1.3), (1.2) is stable when and only when the poles $\chi(\lambda)$ lie in the open left half-plane and the eigenvalues R in modulus are smaller than unity.

Proof. Lemma 1 states the equivalence of Theorems 1 and 2. It will be shown that numbers ρ_{lm}, which are defined by relationship (2.2), can be computed from Eqs. (2.3), (2.4), i.e. we shall prove the following statement. Let P-matrix of Hurvitz $H = A(rr^*)$, $\rho = q^*Hq$, $\chi(\lambda) = r^*(P - \lambda I)^{-1} q$ and $\chi(\lambda) = \gamma(\lambda)/\Delta(\lambda)$ is valid. Then the following relationship holds

$$\rho = \lim_{|\lambda|\to\infty} \lambda \frac{\tau(\lambda)}{\Delta(\lambda)}$$

where $\tau(\lambda)$ is the solution of equation

$$\chi(\lambda)\,\chi(-\lambda) = \frac{\tau(-\lambda)}{\Delta(-\lambda)} + \frac{\tau(\lambda)}{\Delta(\lambda)} \tag{2.5}$$

We have the equality

$$-(P^*H + HP) = -[(P - i\omega I)^* H + H(P - i\omega I)] = rr^*$$

Multiplying this equality on the right by $q_{i\omega} = -(P - i\omega I)^{-1} q$ and on the left by $(q_{i\omega})^*$ we obtain $2\,\mathrm{Re}\, q^*Hq_{i\omega} = |\chi(i\omega)|^2$. Consequently, the polynomial $\tau(\lambda)$ such that $\tau(\lambda)/\Delta(\lambda) = q^*Hq_\lambda$ serves as the unique solution of Eq. (2.5). Since $q^*Hq = \lim_{|\lambda|\to\infty} \lambda q^*Hq_\lambda$, Lemma 1 is proven.

Before going to the proof of Theorem 1, we establish the following statement.

Lemma 2. For a square matrix $\zeta = \| \zeta_{ij} \|_{i,j=1}^{k}$ with nonnegative elements the following statements are equivalent:

a) Vectors $\varkappa$ and η exist with positive components, satisfying the equation $(I - \zeta)\varkappa = \eta$.

b) All successive principal minors θ_l $(l = 1,\ldots,k)$ of the matrix $(I_k - \zeta)$ are positive.

c) The eigenvalues of matrix ζ in modulus are smaller than unity.

The equivalence of statements (b) and (c) and the consequence that (a) follows from them is proven in [9]. We shall show how statement (b) follows from (a). The proof will be carried out by induction.

Since

$$\eta_i = (1 - \zeta_{ii})\varkappa_i - \sum_{j\neq i,\, j=1}^{k} \zeta_{ij}\varkappa_j$$

then $(1 - \zeta_{ii}) \geqslant \eta_i/\varkappa_i > 0$ and, consequently, $\theta_1 > 0$.

Let us assume that $\theta_l > 0$ $(l = 1,\ldots, m;\ m < k)$. We denote $U_l = \| \zeta_{ij} \|_{i,j=1}^{l}$, $f = \| \varkappa_i \|_{i=1}^{m}$. The matrix $(I_{m+1} - U_{m+1})$ can be written in the form

$$(I_{m+1} - U_{m+1}) = \begin{Vmatrix} (I_m - U_m) & -b \\ -c^* & d \end{Vmatrix}$$

where the value $d > 0$, and the m dimensional vectors b and c have nonnegative components. Components of vector $g = (I_m - U_m)f - b\varkappa_{m+1}$ are not smaller than the corresponding components of vector η and, consequently, they are positive. Since the matrix $(I_m - U_m)^{-1}$ has nonnegative components, then

$$-c^*f \leqslant -c^*(I_m - U_m)^{-1} b\varkappa_{m+1}$$
$$0 < \eta_{m+1} \leqslant -c^*f + d\varkappa_{m+1} \leqslant \varkappa_{m+1}(d - c^*(I_m - U_m)^{-1} b)$$

From this we conclude that $\theta_{m+1} = \theta_m (d - c^*(I_m - U_m)^{-1} b) > 0$. Lemma 2 is proven.

Let us prove Theorem 1. For stability of system (1.1), (1.2) it is necessary and sufficient (see [3], chapter 6) that positive definite quadratic forms $V(x) = x^*Hx$ and $W(x) = x^*Gx$ exist and satisfy the following relationship [3]:

$$LV(x) = -W(x) \tag{2.6}$$

$$LV(x) = 2x^*HPx + \sum_{l=1}^{k} \alpha_l^2 x^* r_l q_l^* H q_l r_l^* x$$

Relationship (2.6) is equivalent to the matrix relationship

$$-(P^*H + HP) - \sum_{l=1}^{k} \alpha_l^2 q_l^* H q_l r_l r_l^* = G \tag{2.7}$$

Applying the operator A, which is defined by (2.1) to both parts of (2.7), we obtain

$$H - \sum_{l=1}^{k} \alpha_l^2 q_l^* H q_l A(r_l r_l^*) = A(G) \tag{2.8}$$

It will be shown that the necessary and sufficient condition for the existence of $H > 0$ and $G > 0$ satisfying (2.8), is the fulfillment of statement (a) of Lemma 2 for $\zeta = R^*$, where R is determined in (2.2). This will also accomplish the proof of Theorem 1.

N e c e s s i t y. We denote

$$\varkappa = \| q_l^* H q_l \|_{l=1}^{k}, \qquad \eta = \| q_l^* A(G) q_l \|_{l=1}^{k} \tag{2.9}$$

It follows from (2.8) that $\varkappa$ and η satisfy the linear equations

$$\varkappa_m - \sum_{l=1}^{k} \alpha_l^2 \rho_{lm} \varkappa_l = \eta_m \tag{2.10}$$

S u f f i c i e n c y. It follows from statement (c) of Lemma 2 that the solution of linear system (2.10) has positive components for any vector η with positive components. Let us take any positive definite matrix G. Let us determine the vector η with the aid of the second equation (2.9), and the matrix H from the equation

$$H = A(G) + \sum_{l=1}^{k} \alpha_l^2 \varkappa_l A(r_l r_l^*) > 0$$

Here $\varkappa$ is the solution of (2.10). It is easy to verify that the first equation (2.9) is satisfied. Consequently, H satisfies Eq. (2.8).

N o t e 1. Let the poles of the transfer matrix $\chi(\lambda)$ lie in the open left half-space. Let us denote by $\theta_1, \theta_2, \ldots, \theta_k$ successive principal minors of matrix $(I_k - R)$. Then in the intensity space $\alpha_1, \alpha_2,\ldots,\alpha_k$ the region of stability of system (1.3), (1.2) is determined by the inequalities

$$\theta_1 > 0, \qquad \theta_2 > 0, \ldots, \qquad \theta_k > 0 \tag{2.11}$$

The critical vectors of intensities α° form the boundary of the stability region and satisfy the equation $\theta_k = 0$. In the case of a single-parameter perturbation the critical intensity α° of the system is calculated from the equation $|\alpha^\circ| = \rho^{-1/2}$, where ρ is determined from (2.3), (2.4).

Proof. It is sufficient to prove that vectors of critical intensities α° satisfy the equation $\theta_k = \det(I_k - R) = 0$. Let $R = R^\circ$ be the corresponding matrix to vector α° Then for any $\varepsilon (0 \leqslant \varepsilon < 1)$ the moduli of eigenvalues of matrix εR are less than unity. On the basis of continuity the maximum eigenvalue of matrix R does not exceed unity. However, since α° is the critical vector, the maximum eigenvalue is equal to unity. Consequently, $\theta_k = 0$.

Algorithm for calculation of matrix R from equations (2.3), (2.4). Let the following rational function be given

$$\chi(\lambda) = \frac{\gamma_1\lambda^{\nu-1} + \gamma_2\lambda^{\nu-2} + \ldots + \gamma_\nu}{\lambda^\nu + \Delta_1\lambda^{\nu-1} + \ldots + \Delta_\nu} = \frac{\gamma(\lambda)}{\Delta(\lambda)}$$

Let us assume that

$$\chi(\lambda)\,\chi(-\lambda) = \frac{\tau(\lambda)}{\Delta(\lambda)} + \frac{\tau(-\lambda)}{\Delta(-\lambda)}, \qquad \rho = \lim_{|\lambda|\to\infty} \lambda \frac{\tau(\lambda)}{\Delta(\lambda)}$$

$$\tau(\lambda) = \tau_1\lambda^{\nu-1} + \ldots + \tau_\nu$$

where $\Delta(\lambda)$ is the Hurvitz polynomial. Then the algorithm for the computation of ρ consists of the following operations:

1) Computation of vector δ using the formula:

$$\delta = \begin{Vmatrix} \gamma_1 & 0 & 0 & \ldots & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 & \ldots & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & 0 & \ldots & \gamma_\nu \end{Vmatrix} \begin{Vmatrix} (-1)^{\nu-1}\gamma_1 \\ (-1)^{\nu-2}\gamma_2 \\ - - - - \\ - - - - \\ \gamma_\nu \end{Vmatrix}$$

2) Computation of the first element φ_1 of vector φ, which satisfies the linear algebraic system $H_\nu\varphi = \delta$, where

$$H_\nu = \begin{Vmatrix} \Delta_1 & 1 & 0 & \ldots & 0 \\ \Delta_3 & \Delta_2 & \Delta_1 & \ldots & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & 0 & \ldots & \Delta_\nu \end{Vmatrix}$$

3) Computation of the desired quantity $\rho = {}^1/_2(-1)^{\nu-1}\varphi_1$.

3. Some results of application of Theorem 2 to concrete systems. Stability conditions are presented for a fourth order scalar equation with constant coefficients in the case of stochastic perturbation of one of the coefficients of the equation.

Let the trivial solution of the equation

$$y^{(4)} + ay^{(3)} + by^{(2)} + cy^{(1)} + dy + \alpha\xi^{\cdot}\sigma = 0$$

be asymptotically stable for zero noise ($\alpha = 0$). The conditions of stability are as follows:

$$\sigma = y: \quad \alpha^2 < \frac{d\beta}{ab - c}, \qquad \sigma = y^{(2)}: \quad \alpha^2 < \beta/c$$

$$\sigma = y^{(1)}: \quad \alpha^2 < \beta/a, \qquad \sigma = y^{(3)}: \quad \alpha^2 < \frac{\beta}{bc - ad}$$

$$\beta = 2(abc - c^2 - a^2d)$$

Stability conditions of equations of the second and third orders can be found in [3]. We note that the stability criterion in this paper as applied to linear equations of the νth order is close to the criterion of paper [6].

Stability conditions are presented for a system of two equations of the second order

$$y^{(2)} + ay^{(1)} + bz^{(1)} + cy + dz + \alpha\xi^{\cdot}\sigma = 0$$
$$z^{(2)} + ey^{(1)} + fz^{(1)} + gy + hz = 0$$

We denote $k = a + f$, $l = h + c - be + af$, $m = ah + cf - bg - de$, $n = ch - dg$. Let conditions of asymptotic stability be satisfied for zero noise ($\alpha = 0$): $kl - m > 0$, $klm - k^2n - m^2 > 0$, $n > 0$. The conditions of stability are as follows:

$$\sigma = y: \alpha^2 < \gamma n\{h^2(kl - m) + n[m + (f^2 - 2h)k]\}^{-1}$$
$$\sigma = y^{(1)}: \alpha^2 < \gamma[(n^2 - n)k + (l + f^2 - 2h)m]^{-1}$$
$$\sigma = z: \alpha^2 < \gamma n[g^2(kl - m) + kne^2]^{-1}$$
$$\sigma = z^{(1)}: \alpha^2 < \gamma(g^2k + e^2m)^{-1}$$
$$\gamma = 2(klm - k^2h - m^2)^1$$

The following example is interesting because here the stability region is constructed for a system which is subject to the action of two independent noises of intensities α_1 and α_2. Let us examine the system

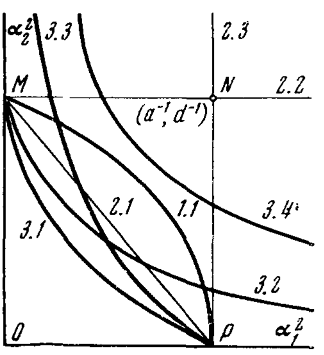

Fig. 1.

$$\sigma_1 = \frac{1}{\Delta(p)}[(a_1p + a_2)\sigma_1\alpha_1\xi_1^{\cdot} + (b_1p + b_2)\sigma_2\alpha_2\xi_2^{\cdot}]$$

$$\sigma_2 = \frac{1}{\Delta(p)}[(c_1p + c_2)\sigma_1\alpha_1\xi_1^{\cdot} + (d_1p + d_2)\sigma_2\alpha_2\xi_2^{\cdot}]$$

$$\Delta(p) = p^2 + \Delta_1 p + \Delta_2$$

here p is the operator of differentiation, $\Delta(p)$ is the Hurvitz polynomial. According to Note 1 the stability region of the system is defined by the inequalities

$$\theta_1 = 1 - \alpha_1^2 a > 0, \ \theta_2 = 1 - \alpha_1^2 a - \alpha_2^2 d + \alpha_1^2\alpha_2^2(ad - bc) > 0$$

$$a = \frac{a_1^2\Delta_2 + a_2^2}{2\Delta_1\Delta_2}, \qquad b = \frac{b_1^2\Delta_2 + b_2^2}{2\Delta_1\Delta_2}$$

$$c = \frac{c_1^2, \Delta_2 + c_2^2}{2\Delta_1\Delta_2}, \qquad d = \frac{d_1^2\Delta_2 + d_2^2}{2\Delta_1\Delta_2}$$

In Fig. 1 boundaries of stability regions for various cases are given. We present the list of examined cases according to the numbers of lines in the Figure (regions of stability in the α_1^2, α_2^2 plane are bounded by coordinate axes and the indicated lines).

1. $ad - bc > 0$: 1.1. $abcd > 0$; 1.2. $bc = 0$, $ad > 0$ (segments MN and NP)
2. $ad - bc = 0$: 2.1. $abcd > 0$; 2.2. $a = b = c = 0$, $d > 0$; 2.3. $b = c = d = 0$, $a > 0$
3. $ad - bc < 0$: 3.1. $abcd > 0$; 3.2. $a = 0$, $bcd > 0$; 3.3. $abc > 0$, $d = 0$; 3.4. $a = d = 0$, $bc > 0$

BIBLIOGRAPHY

1. Kats, I. Ia. and Krasovskii, N. N., On the stability of systems with random parameters. PMM Vol. 24, №5, 1960.
2. Bertram, J. E. and Sarachik, P. E., Stability of circuits with randomly timevarying parameters. Proc. Intern. Sympos. on Circuit and Inform. Theory IRE Transactions CT-6, Los Angeles, Calif., 1959.
3. Khas'minskii, R. Z., Stability of Systems of Differential Equations under Random Perturbations of their Parameters. M., "Nauka", 1969.
4. Nevel'son, M. B. and Khas'minskii, R. Z., On stability of stochastic systems. Problems of Information Transmission, Vol. 2, №3, 1968.
5. Nevel'son, M. B., Some remarks concerning the stability of a linear stochastic system. PMM Vol. 30, №6, 1966.
6. Nevel'son, M. B. and Khasminskii, R. Z., Stability of a linear system with random disturbances of its parameters. PMM Vol. 30, №2, 1966.
7. Kushner, H. J., Stochastic Stability and Control, New York, Academic Press, 1967.
8. Gikhman, I. I., On stability of solutions of stochastic differential equations. In Collection: Limit Theorems and Statistical Derivations. Tashkent "FAN", 1966.
9. Gantmakher, F. R., Theory of Matrices, 3rd ed. M., "Nauka", 1967.

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